44′ 0″, and $\theta_\infty = 90° + D_\infty$. The data in the last four rows fulfill the expectation that the curved ray has sensibly reached its asymptote not only for the earth's orbit but even for an observer at the distance of the inferior planet.[1]

GRAPHICAL DISCUSSION OF THE ROOTS OF A QUARTIC EQUATION.

By E. L. REES, University of Kentucky.

It is the purpose of this note to give a graphical study of the conditions which determine the nature of the roots of a quartic equation. Using the reduced form $f(x) = x^4 + qx^2 + rx + s = 0$, with q, r and s real and with discriminant Δ, we have types of the quartic for which the following are criteria[2] regarding the nature of the roots:

$\Delta < 0$, roots distinct, two real, two imaginary;

$\Delta > 0$, roots distinct, all real or all imaginary;

$$q < 0,\ s > \frac{q^2}{4}, \text{ roots imaginary;}$$

$$s < \frac{q^2}{4}, \text{ roots real;}$$

$$q \geqq 0, \text{ roots imaginary;}$$

$\Delta = 0$, at least two equal roots;

$$q < 0,\ s > \frac{q^2}{4}, \text{ two equal real roots, two imaginary;}$$

$$-\frac{q^2}{12} < s < \frac{q^2}{4}, \text{ roots real, two and only two equal;}$$

$$s = \frac{q^2}{4}, \text{ two pairs of equal real roots;}$$

$$s = -\frac{q^2}{12}, \text{ roots real, three equal;}$$

$$q > 0,\ s > 0,\ r \neq 0, \text{ two equal real roots, two imaginary;}$$

$$s = \frac{q^2}{4},\ r = 0, \text{ two pairs of equal imaginary roots;}$$

$$s = 0, \text{ two equal real roots, two imaginary;}$$

$$q = 0,\ s > 0, \text{ two equal real roots, two imaginary;}$$

$$s = 0, \text{ four equal real roots.}$$

The discriminant is the product of the squares of the differences of the roots

[1] The deviation of a ray from its original direction at the star is obtained by adding $2m/p_0$ to the value of D given in the table, and the total deviation, 1.745″, from asymptote to asymptote is twice the value of D given in the last two lines of the table.—EDITORS.

[2] Compare L. E. Dickson, *Elementary Theory of Equations*, 1914, p. 45.

of $f(x) = 0$. We prove first the following theorem[1] which gives a geometric interpretation of the discriminant that will be used later in the discussion.

The discriminant of a real quartic equation $f(x) = 0$ with leading coefficient unity equals 256 times the product of the ordinates of the turning points[2] *of the graph of $y = f(x)$.*

Denoting by α_i or α_j the roots of $f(x) = 0$, and by α_k' those of $f'(x) = 0$, we have

$$\begin{aligned}\Delta &= \prod_i \prod_{j>i} (\alpha_i - \alpha_j)^2 = \prod_i \prod_{j \neq i} (\alpha_i - \alpha_j) = \prod_i f'(\alpha_i) \\ &= 4^4 \prod_i \prod_k (\alpha_i - \alpha_k') = 4^4 \prod_k \prod_i (\alpha_k' - \alpha_i) = 4^4 \prod_k f(\alpha_k').\end{aligned}$$

Since $f(\alpha_k')$ are the ordinates of the turning points of $y = f(x)$, the proof is complete. We may now state as a corollary the following results:

$\Delta < 0$, *ordinates all negative;*
one ordinate negative, two positive;
one ordinate negative, two imaginary;
$\Delta > 0$, *ordinates all positive;*
one ordinate positive, two negative;
one ordinate positive, two imaginary;
$\Delta = 0$, *at least one ordinate zero.*

Assume the graph of $y = f(x)$ drawn. Let a line start in the position of the x-axis and revolve about the origin into the position of the line $y = -rx$, and let each point of the graph move in a vertical line at such a rate that its distance measured vertically from the revolving line remains constant.[3] The resulting curve is symmetric with respect to the y-axis and is the graph of the equation $y = x^4 + qx^2 + s$, which we shall call the auxiliary quartic. The roots of $f(x) = 0$ are the abscissas of the points of intersection of the line and this curve. The turning points of the original curve correspond to those points of the transformed curve at which the tangent is parallel to the line $y = -rx$. We shall call these points of the graph of the auxiliary quartic *transformed turning points.* The inflection points of one curve correspond to the inflection points of the other curve. The proofs of these statements are quite simple and are left to the reader.

By the usual calculus method we easily deduce the following facts concerning the auxiliary quartic curve.

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[1] The corresponding theorem for the n-ic $x^n + c_1x^{n-1} + \cdots + c_n = 0$ (c's real) is

$$\Delta = (-1)^{\frac{n(n-1)}{2}} n^n \cdot \text{(product of ordinates of turning points)}.$$

[2] In this discussion we shall understand that a point of inflection with horizontal tangent is to be considered a multiple turning point.

[3] This of course is the same as adding the ordinates of the line $y = -rx$ and the curve $y = f(x)$. The effect of this transformation in our discussion is to replace the original quartic by the auxiliary quartic curve while the line $y = -rx$ in its relation to the graph of the auxiliary quartic in a certain sense replaces the x-axis.

The three turning points[1] are $(0, s)$, $\left(\pm\sqrt{-\frac{q}{2}},\, s-\frac{q^2}{4}\right)$ and the two inflection points are $\left(\pm\sqrt{-\frac{q}{6}},\, s-\frac{5}{36}q^2\right)$.

If $q < 0$, these turning points and inflection points are all real and distinct.

The difference between the ordinate of the middle turning point and the common ordinate of the other two is $q^2/4$.

The distance between the y-intercept point of the curve and that of the inflection tangents $= q^2/12$.

The y-intercept of the inflection tangents is $s + q^2/12 = I$.

If $q = 0$, the y-intercept point of the curve is a triple turning point (double inflection point).

If $q > 0$, there is only one real turning point; the inflection tangents are imaginary and meet inside the curve at the point $(0, I)$; and there is a real double tangent $y = s - q^2/4$, with conjugate imaginary points of contact.

We shall study the various cases corresponding to the different forms and positions of the auxiliary quartic curve according to the sign of q and the value of s.

The corollary of the theorem on discriminants proved above may now be restated in a slightly different and more useful form, namely:

At least one and possibly three real transformed turning points lie above or below the line $y = -rx$ according as Δ is greater than or less than 0; and at least one transformed turning point (real with one exception[2]) lies on the line if $\Delta = 0$.

Bearing this corollary in mind, an examination of a figure in each case will make clear the following classification of quartics, equivalent to that given at the beginning of this paper.

Case I. $q < 0$,

$s > \frac{q^2}{4}$,	$\Delta < 0$, two roots real and distinct;
	$\Delta = 0$, two roots real and equal;
	$\Delta > 0$, no real root;
$s = \frac{q^2}{4}$,	$\Delta < 0$, two roots real and distinct;
	$\Delta = 0$, two pairs of equal real roots;
$0 < s < \frac{q^2}{4}$,	$\Delta < 0$, two roots real and distinct;
$s = 0$,	$\Delta = 0$, all roots real, two equal;
$-\frac{q^2}{12} < s < 0$,	$\Delta > 0$, all roots real and distinct;

[1] Not the "transformed turning points" mentioned in the preceding paragraph, but the turning points of the auxiliary quartic itself.

[2] It will be seen (next foot-note) that there is one case when $\Delta = 0$ in which a real turning point (the other two being imaginary) is not on the line.

$s = -\frac{q^2}{12}$, $\Delta < 0$, two roots real and distinct;

$\Delta = 0$, all roots real, three equal;

$s < -\frac{q^2}{12}$, $\Delta < 0$, two roots real and distinct.

Case II. $q = 0$,

$s > 0$, $\Delta < 0$, two roots real and distinct;
$\Delta = 0$, two roots real and equal;
$\Delta > 0$, no real roots;

$s = 0$, $\Delta < 0$, two roots real and distinct;
$\Delta = 0$, four equal real roots;

$s < 0$, $\Delta < 0$, two roots real and distinct.

Case III. $q > 0$,

$s > 0$, $\Delta < 0$, two roots real and distinct;
$\Delta = 0$, two roots real and equal;[1]
$\Delta > 0$, no real root;

$s = 0$, $\Delta < 0$, two roots real and distinct;
$\Delta = 0$, two roots real and equal;

$s < 0$, $\Delta < 0$, two roots real and distinct.

If $I < 0$, then $s < -q^2/12$ and the quartic has two real and distinct and two imaginary roots.

For a triple root it is necessary and sufficient that the line $y = -rx$ be an inflection tangent. I being the y-intercept of the inflection tangents, it follows that $I = 0$ is a necessary condition for a triple root; and since the slopes of the inflection tangents are $\pm \frac{4q}{3}\sqrt{-\frac{q}{6}}$ one of which must equal $-r$, it results that $8q^3 + 27r^2 = 0$ is also a necessary condition.

Conversely, if $I = 0$ and $8q^3 + 27r^2 = 0$, the quartic has a triple root.

Noting that

$$J\left(= \frac{qs}{6} - \frac{r^2}{16} - \frac{q^3}{216}\right) = \frac{q}{6} I - \frac{8q^3 + 27r^2}{432}$$

we see that these results are equivalent to the following familiar theorem:

A necessary and sufficient condition that the quartic equation have three or more roots equal is $I = J = 0$.[2]

The geometric arguments for most of the cases of the theorem may be made without the use of the derivative. For this purpose we apply the transformation

[1] Except when $s = q^2/4$ and $r = 0$, in which case there are two conjugate imaginary double roots.

[2] The method of this paper enables us also to recognize the order of succession of the simple and multiple roots. Thus by noting the double root cases in which the inflexions of the transformed quartic both lie below the line $y = -rx$ we find as a necessary and sufficient condition for a double root separating two simple real roots $\Delta = 0$, $8q^3 + 27r^2 < 0$.

In the other cases where all roots are real and there is one double or triple root, this will be the greatest or least root according as r is positive or negative.—EDITOR.

$x' = x^2$, $y = y$ to the parabola $y = x'^2 + qx' + s$, thus obtaining the auxiliary quartic curve. Note that the vertex of the parabola is $(-q/2, s - q^2/4)$ and that the parabola and quartic have in common the y-intercept s.

The different forms of the quartic curve depend on the position of the parabola relative to the y-axis. The three cases follow.

$q < 0$, the vertex of the parabola is to right of the y-axis; the quartic has three real and distinct turning points, $(0, s)$, $\left(\pm\sqrt{-\frac{q}{2}}, s - \frac{q^2}{4}\right)$.

$q = 0$, the vertex of the parabola is on the y-axis; the quartic has a triple turning point $(0, s)$.

$q > 0$, the vertex of the parabola is to left of the y-axis; the quartic has only one real turning point $(0, s)$.

The arguments for the various cases of the theorem with few exceptions are identical with those sketched above.[1]

TWO NEW CONSTRUCTIONS OF THE STROPHOID.

By R. M. MATHEWS, Wesleyan University.

(Read before the American Mathematical Society December 28, 1920.)

1. The classic construction for the strophoid uses a pencil of circles each of which has its center on a "medial" line g and passes through a fixed point, the node O, on g (Fig. 1). Let each circle be cut by that one of its diameters which passes through a fixed point, the singular focus F. The curve is the locus of these intersections.[2] The object of this note is to make this construction more general for the same curve: first, by using any line through the node as locus for the centers of the circles; and second, by using a pencil of circles through any two conjugate points of the curve. In preparation for this we describe certain well known features of the curve.[3]

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[1] Instead of adding the ordinates of the line $y = -rx$ and the curve $y = f(x)$, the author might have started with the curve $y = x^4 + qx^2 + s$ and regarded the roots of the given quartic as the abscissas of the intersections of this curve and the line $y = -rx$. The form of this curve depends only on q; its position, or the position of the origin with respect to it, depends on s, while the character of the roots of the equation, when q and s are given, depends on r. Thus the classification, based first on q, and then on s, would finally be based on r.

The range of values of r for any type of equation, when q and s are given, depends on those values which correspond to the real tangents from the origin. These values of r are the roots of the equation $\Delta = 0$, and for any particular type of equation Δ will have a particular sign or be zero. Conversely, the sign or vanishing of Δ, with the given values of q and s, will usually determine the type of the equation. These considerations would enable us to dispense with the author's theorem on discriminants. Results obtained as depending on r could be interpreted at once as depending on Δ, and so when the classification has been obtained, the various classes could be grouped and arranged with respect to Δ, q and s if such an arrangement is more convenient for use.—EDITOR.

[2] Gino Loria, *Spezielle algebraische und transcendente ebene Kurven*, volume 1, Leipzig, 1910, p. 60. The strophoid of our text-books is the *right* strophoid, the form this curve takes when the node is at the foot of the perpendicular from the focus to the median.

[3] Loria, *loc. cit.*, chapter 8.